

SUBELLIPTIC ESTIMATES FOR THE $\bar{\partial}$ -PROBLEM ON A SINGULAR COMPLEX SPACE

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ABSTRACT. We prove subelliptic estimates for the $\bar{\partial}$ -problem at the isolated singularity of the variety $z^2 = xy$ in \mathbb{C}^3 .

1. INTRODUCTION

The Cauchy-Riemann operator $\bar{\partial}$ and the related $\bar{\partial}$ -Neumann operator play a central role in complex analysis. Especially the L^2 -theory for these operators is of particular importance and has become indispensable for the subject after the fundamental work of Hörmander on L^2 -estimates and existence theorems for the $\bar{\partial}$ -operator [H] and the related work of Andreotti and Vesentini [AV]. By no means less important is Kohn's solution of the $\bar{\partial}$ -Neumann problem, which implies existence and regularity results for the $\bar{\partial}$ -complex, as well (see [FK]). Important applications of the L^2 -theory are e.g. the Ohsawa-Takegoshi extension theorem [OT], Siu's analyticity of the level sets of Lelong numbers [S1] or the invariance of plurigenera [S2] – just to name some.

Whereas the theory of the $\bar{\partial}$ -operator and the $\bar{\partial}$ -Neumann operator is very well developed on complex manifolds, still not too much is known about the situation on singular complex spaces which appear naturally as the zero sets of holomorphic functions. The further development of this theory is an important endeavor since analytic methods have led to fundamental advances in geometry on complex manifolds, but these analytic tools are still missing on singular spaces.

The topic has attracted some attention recently and considerable progress has been made. Let us mention e.g. the development of some Koppelman formulas by Andersson and Samuelsson ([AS1], [AS2]). Concerning the L^2 -theory for the $\bar{\partial}$ -operator, Øvrelid and Vassiliadou obtained essential results for the case of isolated singularities. Following a path prepared by Fornæss, Diederich, Vassiliadou and Øvrelid ([F], [DFV], [FOV], [OV1], [OV2]) and by Ruppenthal and Zeron (see [R1], [R2], [R3], [RZ1], [RZ2]), they were finally able to describe the L^2 -cohomology for the $\bar{\partial}$ -operator at isolated singularities completely in terms of a resolution of singularities (see [OV3]). For another, different approach to these results we refer also to [R5].

These works mark important progress concerning the understanding of the obstructions to solving the $\bar{\partial}$ -equation at isolated singularities. It remains to study the regularity of the equation: on domains in complex manifolds, the close connection between the regularity of the $\bar{\partial}$ -equation on one hand and the geometry of the domain (and its boundary) on the other hand is one of the central topics of complex analysis. Recall that a domain in \mathbb{C}^n with smooth pseudoconvex boundary is of finite type exactly if the $\bar{\partial}$ -Neumann problem is subelliptic, and that there is a deep

connection between the type of the boundary and the order of subellipticity (cf. the works of Kohn, Catlin and D'Angelo). It is an interesting endeavor to establish such connections also between the regularity of the $\bar{\partial}$ -equation at singularities and the geometry of the singularities.

Compactness of the $\bar{\partial}$ -Neumann operator, which can be seen as a boundary case of subelliptic regularity, has been established at isolated singularities recently in [R4] and [OR]. Compactness of the $\bar{\partial}$ -Neumann operator is an important property in the study of weakly pseudoconvex domains. Moreover, it yields that the corresponding space of L^2 -forms has an orthonormal basis consisting of eigenforms of the $\bar{\partial}$ -Laplacian $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. The eigenvalues of \square are non-negative, have no finite limit point and appear with finite multiplicity.

In the present paper, we make the next step and study subelliptic estimates for the $\bar{\partial}$ -problem at an isolated singularity. Besides the general question whether it is possible to classify singularities by the degree of subelliptic estimates which hold, the main motivation is as follows. It is one of the key observations in [FOV], [R2], [OV3], [R5] that the $\bar{\partial}$ -equation can be solved in the L^2 -category at isolated singularities with some gain of regularity, and one reason why the theory is not well developed for arbitrary singularities is that such results do not yet exist in that case (one can only solve in the L^2 -category but something better is needed). In view of such questions it is natural to consider the canonical $\bar{\partial}$ -solution operator $\bar{\partial}^*N$, where N is the $\bar{\partial}$ -Neumann operator, and to study regularity of N and $\bar{\partial}^*N$.

To begin with, it makes sense to consider a simple example. So, we decided to study the isolated singularity of the variety $Z := \{z_3^2 = z_1z_2\}$ in \mathbb{C}^3 . Let X be the intersection of Z with the cylinder $\{|z_1|^2 + |z_2|^2 < 1\}$ in \mathbb{C}^3 . We note that X has a strictly pseudoconvex boundary and consider X as a Hermitian complex space with the restriction of the Euclidean metric of \mathbb{C}^3 . We define the weighted L^2 -spaces of (p, q) -forms on X :

$$L_{(p,q)}^{2,k}(X) = \left\{ f : \|f\|_{L^{2,k}}^2 = \int_X \gamma^{2k} |f|^2 dV_X < \infty \right\},$$

with the weight

$$\gamma = \sqrt{|z_1| + |z_2|}.$$

Note that $\gamma^2 \sim \|z\| = \sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2}$ on X . For ease of notation, we will write simply $\|f\|$ for $\|f\|_{L^{2,0}}$.

We also make use of some Sobolev spaces on X . To introduce these spaces, we consider the 2-sheeted covering $\pi : \mathbb{C}^2 \rightarrow Z$, $(v, w) \mapsto (z_1, z_2, z_3) = (v^2, w^2, vw)$ for which $\pi^*dV_X \sim (\pi^*\gamma^4)dV_{\mathbb{C}^2}$. We say that a function f is in $W^k(X)$ if all the partial derivatives up to order k of π^*f with respect to the (v, w) -coordinates are square-integrable with respect to the volume $(\pi^*\gamma^4)dV_{\mathbb{C}^2}$ on \mathbb{C}^2 (see (5.2)). Non-integer Sobolev spaces $W^\epsilon(X)$ are then defined as usually by use of the Fourier transform on \mathbb{C}^2 (see (5.3)). For differential forms, we define the W^ϵ -norm as the sum of the W^ϵ -norms of its coefficients in an orthonormal coordinate system.

Our first main result is as follows:

Theorem 1.1. *Let $f \in L_{(0,1)}^{2,-1}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with support in a neighborhood of the singularity. Then we have the estimate*

$$\|f\|_{W^1(X)}^2 \lesssim \|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2 + \|f\|_{L^{2,-1}}^2.$$

Theorem 1.1 is a special case of our more general estimate (5.1). Note that particularly bounded forms are in $L^{2,-2}(X)$ as $\|z\|^{-2}$ is integrable on X , i.e. $\|f\|_{L^{2,-2}} \lesssim \|f\|_{L^\infty}$. Theorem 1.1 shows that we can have full regularity for forms with a certain extra-vanishing at the singularity. This makes sense as it is known that there exist usually finitely many obstructions to solving the $\bar{\partial}$ -equation in the L^2 -category at isolated singularities and that the number of obstructions to solving $\bar{\partial}u = f$ is decreasing if f is in L^p for increasing p (see [R2], Theorem 1.1 and Theorem 1.2).

In this spirit, we derive a type of subelliptic estimates for the $\bar{\partial}$ -problem which can be viewed as a trade-off of a decrease in derivatives for less stringent hypotheses on to which L^p spaces a given form may belong. Our results show that the restriction of a form f from more L^p spaces for $p > 2$ allows for higher orders of derivatives of f to be estimated.

Then our second main result reads as:

Theorem 1.2. *Let $0 \leq \epsilon \leq 1/2$, $p > 4/(2 - \epsilon)$, and $f \in L^p_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. Then the following subelliptic estimate holds*

$$\|f\|_{W^\epsilon(X)} \lesssim \|\bar{\partial}f\| + \|\bar{\partial}^*f\| + \|f\|_{L^p(X)}.$$

Here, the restriction to $\epsilon \leq 1/2$ is only due to the fact that X has a strongly pseudoconvex boundary. For forms with compact support in X the statement is valid for all $0 \leq \epsilon \leq 1$, by Theorem 1.1.

2. PRELIMINARY CALCULATIONS

We note that the variety X can be parametrized by the two sheeted covering

$$\pi : \mathbb{C}^2 \rightarrow X, (v, w) \mapsto (v^2, w^2, vw).$$

Instead of working directly on X , we consider instead the strongly pseudoconvex domain $B = \{(v, w) : |v|^4 + |w|^4 < 1\}$ in \mathbb{C}^2 with the metric

$$(2.1) \quad (g_{ij}) = \begin{pmatrix} 4|v|^2 + |w|^2 & v\bar{w} \\ \bar{v}w & |v|^2 + 4|w|^2 \end{pmatrix},$$

so that the volume element is given by

$$(2.2) \quad dV_g = \det(g_{ij}) dv \wedge dw \wedge d\bar{v} \wedge d\bar{w}.$$

Then we have simply

$$\|f\|_{L^{2,k}}^2 = \frac{1}{2} \int_B (\pi^*\gamma)^{2k} |\pi^*f|^2 dV_g$$

for the weighted L^2 -norms on X . So, for the questions that we study in the present paper, it is absolutely sufficient to replace the original variety X in \mathbb{C}^3 by the smooth domain B in \mathbb{C}^2 with the positive semi-definite pseudometric $g = (g_{ij})$.

Thus, from now on, let X be the Hermitian space (B, g) . It is our goal to study subelliptic estimates for the $\bar{\partial}$ -problem on the Hermitian complex space X .

We can calculate $\bar{\partial}$ and $\bar{\partial}^*$ in the holomorphic coordinates v and w (see [M]). For $f \in L^2(X)$ we have

$$\bar{\partial}f = \frac{\partial f}{\partial \bar{v}} d\bar{v} + \frac{\partial f}{\partial \bar{w}} d\bar{w},$$

and in the case $f = f_1 d\bar{v} + f_2 d\bar{w}$, $f \in L^2_{(0,1)}(X)$, we have

$$\bar{\partial}f = \left(\frac{\partial f_2}{\partial \bar{v}} - \frac{\partial f_1}{\partial \bar{w}} \right) d\bar{v} \wedge d\bar{w}.$$

To describe the operator $\bar{\partial}^*$ we first define

$$|g| = \det(g_{ij}) = 16|v|^2|w|^2 + 4|v|^4 + 4|w|^4.$$

Thus we can write

$$(g^{ij}) = \frac{1}{|g|} \begin{pmatrix} |v|^2 + 4|w|^2 & -v\bar{w} \\ -\bar{v}w & 4|v|^2 + |w|^2 \end{pmatrix}.$$

For the operator $\bar{\partial}^*$ acting on a $(0,1)$ form, $f = f_1 d\bar{v} + f_2 d\bar{w}$ we can write

$$\bar{\partial}^* f = \frac{1}{|g|} \left(\frac{\partial}{\partial v} |g| g^{11} f_1 + \frac{\partial}{\partial w} |g| g^{12} f_1 + \frac{\partial}{\partial v} |g| g^{21} f_2 + \frac{\partial}{\partial w} |g| g^{22} f_2 \right).$$

From the discussion above, we immediately see that singularities occur in the operators under study at $v = w = 0$. We make our calculations with respect to an orthonormal frame of $(1,0)$ and $(0,1)$ forms and we keep track of the singularities arising at $v = w = 0$ in the use of such forms. As we shall see, we need to consider fields with coefficients with singular behavior near the singularity of X . We keep track of the blow up near the singularity in terms of the factor

$$\gamma = \sqrt{|v|^2 + |w|^2}.$$

Note that γ^2 behaves like the distance to the origin in X (with respect to the metric g), because $\gamma^2 = |z_1| + |z_2|$ when we consider γ in the z -coordinates on the original variety in \mathbb{C}^3 carrying the restriction of the Euclidean metric.

A simple calculation shows an orthonormal system of $(1,0)$ forms will consist of forms written as

$$\alpha_1(v, w)dv + \alpha_2(v, w)dw,$$

where

$$|\alpha_i| \sim \gamma \quad i = 1, 2$$

and hence the dual frame consists of vectors which can be written as

$$(2.3) \quad \beta_1(v, w) \frac{\partial}{\partial v} + \beta_2(v, w) \frac{\partial}{\partial w},$$

where

$$|\beta_i| \sim \frac{1}{\gamma} \quad i = 1, 2.$$

We follow the notation in [EL] to keep track of the singularity near the origin. We write ξ_k to denote an operator which on the level of functions is the multiplication by a function $\xi_k(\zeta)$ with the property

$$(2.4) \quad |\gamma^\alpha D_\zeta^\alpha \xi_k(\zeta)| \lesssim \gamma^k,$$

where D_ζ^α is a differential operator, using the index notation, of order $|\alpha|$: for a multi-index $\alpha = (\alpha_1, \alpha_2)$, where each α_j is an integer, we use the notation

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial \zeta_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \zeta_2^{\alpha_2}}$$

$$|\alpha| = \alpha_1 + \alpha_2.$$

On the level of forms, $\xi_k f$ is a sum of terms which are the product of coefficients of f with forms whose coefficients are of type ξ_k on the level of functions. Thus, for instance with $f = f_1 \omega_1 + f_2 \omega_2$, $\xi_k f$ could be used to denote a $(0,1)$ -form such as $\xi_k^1 f_1 \omega_1 + \xi_k^2 f_2 \omega_2$, where ξ_k^1 and ξ_k^2 satisfy estimates as in (2.4), or $\xi_k f$ could be used to denote a function $\xi_k^1 f_1 + \xi_k^2 f_2$.

We choose an orthonormal frame ω_1, ω_2 of $(1,0)$ vectors as above, and denote by L_1 and L_2 the dual frame. In the next section we will see the following two calculations come into play. From the discussion above we compute

$$(2.5) \quad \bar{\partial} \omega^i = \sum_{j,k} \xi_{-2} \bar{\omega}^j \wedge \omega^k$$

$$(2.6) \quad [L_j, \bar{L}_k] = \sum_i \xi_{-2} L_i + \sum_i \xi_{-2} \bar{L}_i.$$

3. INTEGRATION BY PARTS

We take as our guide the situation on a smoothly bounded strictly pseudoconvex domain, $\Omega \subset \subset \mathbb{C}^2$. In such a setting, for $f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2$, where $f_i \in C^1(\bar{\Omega})$, we have the Morrey-Kohn-Hörmander formula (see [CS]):

$$(3.1) \quad \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 = \sum_{i,j=1}^2 \int_{\Omega} \left| \frac{\partial f_i}{\partial \bar{z}_j} \right|^2 dV + \sum_{i,j=1}^2 \int_{\partial\Omega} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} f_j \bar{f}_j dS,$$

where dV refers to the volume element, dS the surface area element, and ρ is a defining function for the domain Ω . An important fact to be used later is that strict pseudoconvexity implies the Levi form associated with the boundary is strictly positive definite and thus allows the last term on the right to be bounded from below. The relation in (3.1) is obtained by an integration by parts in the norms $\|\bar{\partial} f\|^2$ and $\|\bar{\partial}^* f\|^2$. We shall see below that an integration by parts leads to weighted norms, the weights of which blow up at our singularity due to the singular nature of the operators. We consider the norms $\|\bar{\partial} f\|^2$ and $\|\bar{\partial}^* f\|^2$ in terms of the vector fields L_1, L_2, \bar{L}_1 , and \bar{L}_2 .

We write a $(0,1)$ -form, f , as $f = f_1 \bar{\omega}_1 + f_2 \bar{\omega}_2$. For f a $(0,1)$ -form which is $C^1(\mathbb{C}^2)$ (in the sense the derivatives with respect to coordinates given by v and w in Section 2 are continuous) with compact support in a neighborhood of the singularity, we can write, using (2.5),

$$(3.2) \quad \bar{\partial} f = (\bar{L}_1 f_2 - \bar{L}_2 f_1) \bar{\omega}_1 \wedge \bar{\omega}_2 + \xi_{-2} f,$$

and similarly, using integration by parts and the fact that the vector fields, L_1 and L_2 have singular coefficients as in (2.3), we have

$$(3.3) \quad \bar{\partial}^* f = L_1 f_1 + L_2 f_2 + \xi_{-2} f.$$

We define the norm

$$\|\bar{L} f\|^2 = \sum_{i,j=1}^2 \|\bar{L}_j f_i\|^2.$$

Likewise, the norm

$$\|L f\|^2 = \sum_{i,j=1}^2 \|L_j f_i\|^2$$

will be used. Through an integration by parts and the commutator relation (2.6) above the two norms are related:

$$\begin{aligned}
 (L_j f_i, L_j f_i) &= (f_i, \overline{L}_j L_j f_i) + O(\|\xi_{-2} f\| \|L_j f_i\|) \\
 &= (f_i, L_j \overline{L}_j f_i) + O(\|\xi_{-2} f\| (\|L f\| + \|\overline{L} f\|)) \\
 (3.4) \quad &= (\overline{L}_j f_i, \overline{L}_j f_i) + O(\|\xi_{-2} f\| (\|L f\| + \|\overline{L} f\|)).
 \end{aligned}$$

Summing over i, j and using the notation "(s.c.)" for an arbitrarily small constant, we have

$$\|L f\| \lesssim \|\overline{L} f\| + \|\xi_{-2} f\| + (s.c.) (\|L f\| + \|\overline{L} f\|)$$

which implies

$$(3.5) \quad \|L f\| \lesssim \|\overline{L} f\| + \|\xi_{-2} f\|.$$

We insert (3.2) and (3.3) into the calculation of the norms $\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2$ and follow [CS] to write

$$(3.6) \quad \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 = \|\overline{L} f\|^2 - \sum_{j,k} (\overline{L}_k f_j, \overline{L}_j f_k) + \sum_{j,k} (L_j f_j, L_k f_k) + error,$$

where here and below "error" will refer to terms which can be estimated by

$$\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 + (s.c.) (\|\overline{L} f\|^2 + \|L f\|^2) + \|\xi_{-2} f\|^2.$$

"(s.c.)" is the notation for an arbitrarily small constant.

The middle terms on the right hand side of (3.6) can be related as in (3.4):

$$(L_j f_j, L_k f_k) = (\overline{L}_k f_j, \overline{L}_j f_k) + error.$$

Combining we get

$$\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 = \|\overline{L} f\|^2 + error.$$

Thus we obtain the analogue of the Morrey-Kohn-Hörmander estimate applied to $C^1(0, 1)$ forms with compact support in a neighborhood of the singular point:

$$\begin{aligned}
 \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 &\gtrsim \|\overline{L} f\|^2 - (\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 + \|\xi_{-2} f\|^2) - (s.c.) (\|\overline{L} f\|^2 + \|L f\|^2) \\
 &\gtrsim \|\overline{L} f\|^2 - (\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 + \|\xi_{-2} f\|^2) - (s.c.) \|\overline{L} f\|^2 \\
 &\gtrsim \|\overline{L} f\|^2 - (\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 + \|\xi_{-2} f\|^2),
 \end{aligned}$$

where (3.5) is used in the second step. Hence we have the estimate for $\|\overline{L} f\|$:

$$(3.7) \quad \|\overline{L} f\|^2 \lesssim \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 + \|\xi_{-2} f\|^2.$$

So far the analysis which led to the estimate in (3.7) was done with forms with compact support in a neighborhood of the singularity. To globalize such estimates we can take a partition of unity and consider separately those forms with support in a neighborhood of the singularity and those whose support intersects the boundary. For forms which are supported in a neighborhood intersecting the boundary we rely on the condition of strict pseudoconvexity to handle boundary terms. In the case of a form f supported in a neighborhood, U , of a boundary point of ∂X , the estimates can be read directly from the classical estimates of strictly pseudoconvex domains in complex manifolds [CS]: for $f \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \cap C_{(0,1)}^2(U)$, we have

$$\sum_{i,j=1}^2 \int_{U \cap \partial X} \rho_{jk} f_j \overline{f}_k dS + \|\overline{L} f\|^2 \lesssim \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2.$$

Using

$$\sum_{i,j=1}^2 \int_{U \cap \partial X} \rho_{jk} f_j \bar{f}_k dS > \|f\|_{L^2(\partial X)}^2$$

and a density argument we have the estimate for forms in $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ supported in neighborhood of boundary point:

$$(3.8) \quad \|\bar{L}f\|^2 \leq \|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2.$$

Using a partition of unity and the corresponding contributions from (3.7) and (3.8) leads to the

Proposition 3.1. *Let $f \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \cap C_{(0,1)}^1(X)$. Then we have the estimate*

$$\|\bar{L}f\|^2 \leq \|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2 + \|\xi_{-2}f\|^2.$$

Note that

$$\|\xi_{-2}f\|^2 \sim \|f\|_{L^{2,-2}}^2.$$

4. APPROXIMATION BY SMOOTH FORMS

The goal of this section is to relax the condition of Proposition 3.1 so as to apply the proposition to all $f \in L_{(0,1)}^{2,-2}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. Using a partition of unity we will assume in this section that f has support near the origin (approximation of forms supported near the strictly pseudoconvex boundary may be handled as in the classical theory). We thus let $U \subset \mathbb{C}^2$ be a neighborhood of the origin containing the support of f and $x = (x_1, \dots, x_4)$ the real coordinates in U . We take $\chi(x) \in C_0^\infty(U)$ as an approximation of the identity:

$$\int_U \chi(x) d\mathbf{x}^4 = 1,$$

where $d\mathbf{x}^4 = dx_1 \cdots dx_4$. Furthermore,

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^4} \chi\left(\frac{x}{\varepsilon}\right).$$

Our immediate goal for a given $f \in L_{(0,1)}^{2,-2}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ is to find a sequence of C^1 forms to which Proposition 3.1 can be applied and so that in the limit the inequality

$$\|\bar{L}f\|^2 \leq \|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2 + \|\xi_{-2}f\|^2$$

is obtained.

To this end, for $f = f_1 \bar{\omega}_1 + f_2 \bar{\omega}_2$, we define

$$f_\varepsilon = (f_1 * \chi_\varepsilon) \bar{\omega}_1 + (f_2 * \chi_\varepsilon) \bar{\omega}_2,$$

where convolution is taken with respect to the Euclidean volume element on \mathbb{R}^4 , i.e.

$$f_j = \int f_j(x-y) \chi_\varepsilon(y) d\mathbf{y}^4 \quad j = 1, 2.$$

Let D be a differential operator of the form

$$D = \beta(u, v) \frac{\partial}{\partial x_i}$$

such that $\gamma^2 \beta$ is Lipschitz continuous. This is the case e.g. if β is C^1 -smooth outside the origin and $|\beta| \lesssim 1/\gamma$ as with components of our vector fields in (2.3).

Lemma 4.1. *Let $f \in L^{2,-2}(X)$ such that $Df \in L^2(X)$. Then*

$$(4.1) \quad Df_\epsilon \rightarrow Df \quad \text{in} \quad L^2(X).$$

Proof. Let us first recall that for a function g , we have that

$$\|g\|_{L^2(X)} \sim \|\gamma^2 g\|_{L^2(\mathbb{C}^2)},$$

where we denote by $\|\cdot\|_{L^2(\mathbb{C}^2)}$ the standard Euclidean L^2 -norm in \mathbb{C}^2 . More generally, we have

$$\|g\|_{L^{2,k}(X)} = \|\gamma^k g\|_{L^2(X)} \sim \|\gamma^{2+k} g\|_{L^2(\mathbb{C}^2)}$$

for any weight $k \in \mathbb{R}$.

Hence, we note that

$$f, \gamma^2 Df \in L^2(\mathbb{C}^2).$$

Thus f_ϵ is in fact well defined (and smooth).

We also note that (4.1) is equivalent to

$$\gamma^2 Df_\epsilon \rightarrow \gamma^2 Df \quad \text{in} \quad L^2(\mathbb{C}^2).$$

So, it makes sense to study the operator $\gamma^2 D$ which we may write as

$$P := a(u, v) \frac{\partial}{\partial x_i},$$

where a is Lipschitz continuous by assumption. Our problem is reduced to showing that

$$Pf_\epsilon \rightarrow Pf \quad \text{in} \quad L^2(\mathbb{C}^2)$$

for $f \in L^2(\mathbb{C}^2)$ such that $Pf \in L^2(\mathbb{C}^2)$.

But that holds by the well-known Friedrichs extension lemma (see e.g. [CS], Corollary D.2, which holds for Lipschitz continuous coefficients as is immediate from the proof in [CS]). \square

We now apply Lemma 4.1 to $\bar{\partial}f$. Here we work with the differential operator D on a form $g = g_1\omega_1 + g_2\omega_2$ defined by

$$Dg = \bar{L}_1 g_2 - \bar{L}_2 g_1.$$

Then Lemma 4.1 shows that, for $f \in L^{2,-2}_{(0,1)}(X)$, $Df_\epsilon \rightarrow Df$. It is trivial that

$\xi_{-2} f_\epsilon \xrightarrow{L^2} \xi_{-2} f$ for $f \in L^{2,-2}_{(0,1)}(X)$, and so we have $\bar{\partial}f_\epsilon \xrightarrow{L^2} \bar{\partial}f$. Furthermore, the proof of Lemma 4.1 can be applied to the first order differential operator associated with the operator $\bar{\partial}^*$ and so as a corollary, we have

Corollary 4.2. *Let $f \in L^{2,-2}_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. Then*

$$\begin{aligned} \bar{\partial}f_\epsilon &\xrightarrow{L^2} \bar{\partial}f \\ \bar{\partial}^* f_\epsilon &\xrightarrow{L^2} \bar{\partial}^* f. \end{aligned}$$

Given $f \in L^{2,-2}_{(0,1)}(X)$ we can now apply Proposition 3.1 to f_ϵ to obtain

$$\|\bar{L}f_\epsilon\|^2 \leq \|\bar{\partial}f_\epsilon\|^2 + \|\bar{\partial}^* f_\epsilon\|^2 + \|\xi_{-2} f_\epsilon\|^2.$$

Letting $\epsilon \rightarrow 0$ and using Corollary 4.2 shows that for $f \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \cap L^{2,-2}_{(0,1)}(X)$, we have the estimate

$$\|\bar{L}f\|^2 \leq \|\bar{\partial}f\|^2 + \|\bar{\partial}^* f\|^2 + \|\xi_{-2} f\|^2$$

from which we finally conclude the generalization of Proposition 3.1:

Theorem 4.3. *Let $f \in L_{(0,1)}^{2,-2}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. Then we have the estimate*

$$\|\bar{L}f\|^2 \leq \|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2 + \|\xi_{-2}f\|^2.$$

We again note that $\|\xi_{-2}f\|^2 \sim \|f\|_{L^{2,-2}}^2$.

5. INTERMEDIATE SOBOLEV NORMS

For the time being we work on a neighborhood of the singularity. Then Theorem 4.3 can be combined with (3.5),

$$\|Lf\| \lesssim \|\bar{L}f\| + \|\xi_{-2}f\|,$$

to show

$$(5.1) \quad \|Lf\|^2 + \|\bar{L}f\|^2 \lesssim \|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2 + \|\xi_{-2}f\|^2$$

for $f \in L_{(0,1)}^{2,-2}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with support in a neighborhood of the singularity.

We now define Sobolev spaces on our space X . We recall the association of X with the Hermitian space (B, g) from Section 2, where $B = \{|v|^4 + |w|^4 < 1\}$ and $g \sim \gamma^4$ is the metric (2.1). We let $v = x_1 + ix_2$ and $w = x_3 + ix_4$. For k an integer, $W^k(X)$ is the space of functions whose derivatives with respect to the coordinates x_j of order less than or equal to k are in $L^2(X)$. Thus, for $f \in W^k(X)$

$$(5.2) \quad \|f\|_{W^k(X)} \sim \sum_{|l| \leq k} \int_B \left| \frac{\partial^{|l|}}{\partial x^l} f \right|^2 \gamma^4 d\mathbf{x}^4,$$

where $l = (l_1, \dots, l_4)$ is a multi-index of length 4 and

$$\frac{\partial^{|l|}}{\partial x^l} = \frac{\partial^{l_1}}{\partial x_1^{l_1}} \cdots \frac{\partial^{l_4}}{\partial x_4^{l_4}}.$$

For a $(0, 1)$ -form, $f = f_1 \bar{\omega}_1 + f_2 \bar{\omega}_2$, the $W_{(0,1)}^k$ -norm of f is equivalent to the sum of the W^k -norms of f_1 and f_2 .

We note that any derivative in the x coordinates above is a combination (by multiplication by bounded functions) of the vector fields γL_j and $\gamma \bar{L}_j$, for $j = 1, 2$. Given a form $f \in L_{(0,1)}^{2,-1}(X)$, then $\gamma f \in L_{(0,1)}^{2,-2}(X)$, and we can thus apply (5.1) to γf to bound the $W^1(X)$ norm of a $(0, 1)$ form according to

Theorem 5.1. *Let $f \in L_{(0,1)}^{2,-1}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with support in a neighborhood of the singularity. Then we have the estimate*

$$\|f\|_{W^1(X)}^2 \lesssim \|\gamma \bar{\partial}f\|^2 + \|\gamma \bar{\partial}^*f\|^2 + \|\xi_{-1}f\|^2.$$

Our subelliptic estimates can be viewed as a trade-off of a decrease in derivatives for less stringent hypotheses on to which L^p spaces a given form may belong. Our results show that the restriction of a form f from more L^p spaces for $p > 2$, allows for higher orders of derivatives of f to be estimated.

For non-integer Sobolev spaces we use the Fourier multiplier operator on \mathbb{R}^4 with the symbol $(1 + |\zeta|^2)^{\epsilon/2}$. $\Lambda^\epsilon f$ is then defined through its Fourier transform:

$$\widehat{\Lambda^\epsilon f}(\zeta) = (1 + |\zeta|^2)^{\epsilon/2} \widehat{f}(\zeta).$$

$W^\epsilon(X)$ is the restriction to the domain B of the space of functions on \mathbb{R}^4 which satisfy

$$(5.3) \quad \int_{\mathbb{R}^4} |\Lambda^\epsilon f|^2 \gamma^4 d\mathbf{x}^4 < \infty.$$

The rest of this section is dedicated to proving an estimate for a norm for $W^\epsilon(X)$. First we note that for a function supported outside a neighborhood of the singularity at the origin, its weighted L^2 norm is equivalent to an unweighted norm. If we define

$$\Delta_\lambda := \{|v|^4 + |w|^4 < \lambda\}$$

for $\lambda < 1$, then $\|f\|_{W^\epsilon(X \setminus \Delta_\lambda)}$ is equivalent to the usual W^ϵ norm on the domain $B \setminus \Delta_\lambda \subset \mathbb{C}^2$. With $\lambda_1 < \lambda_2 < 1$ we let $\varphi_{\lambda_1, \lambda_2} \in C^\infty(\mathbb{R}^4)$ with support in Δ_{λ_2} and such that $\varphi_{\lambda_1, \lambda_2} \equiv 1$ in Δ_{λ_1} . Then

$$\|f\|_{W^\epsilon(X)} \lesssim \|f\|_{W^\epsilon(X \setminus \Delta_{\lambda_1})} + \|\varphi_{\lambda_1, \lambda_2} f\|_{W^\epsilon(X)},$$

where, from above,

$$\|\varphi_{\lambda_1, \lambda_2} f\|_{W^\epsilon(X)} \sim \int_{\mathbb{R}^4} |\Lambda^\epsilon \varphi_{\lambda_1, \lambda_2} f|^2 \gamma^4 d\mathbf{x}^4.$$

Without further notation we assume until otherwise stated that the functions we work with have support in a neighborhood of the singularity.

In analogy with our weighted L^2 spaces we can define weighted versions of the Sobolev spaces in (5.2):

$$W_{(p,q)}^{s,k}(X) = \left\{ f : \|f\|_{W^{s,k}}^2 = \sum_{|l| \leq s} \int_B \gamma^{2k} \left| \frac{\partial^{|l|}}{\partial x^l} f \right|^2 \gamma^4 d\mathbf{x}^4 < \infty \right\},$$

Given $0 \leq \epsilon \leq 1$, if $f \in W^{1,1-\epsilon}(X)$, we can write

$$(5.4) \quad \begin{aligned} \|f\|_{W^\epsilon(X \setminus \Delta_\lambda)}^2 &\leq C_\lambda \|f\|_{W^1(X \setminus \Delta_\lambda)}^2 \\ &\leq C'_\lambda \|f\|_{W^{1,1-\epsilon}(X)}^2 \end{aligned}$$

with constants $C_\lambda, C'_\lambda > 0$ which do not depend on f .

The idea to obtain a W^ϵ estimate for functions $f \in W^{1,1-\epsilon}(X)$ is to let $\lambda \rightarrow 0$ in (5.4). However, we note that the constant of the inequality in (5.4) may depend on λ . The rest of the section shows that when using (5.4) to estimate $\|f\|_{W^\epsilon(X)}$, the λ -dependence of the constant may be controlled by adjusting an L^2 norm on the right hand side.

For $\alpha < 1/2$ we can use a simple construction to find $\varphi_\alpha = \varphi_{\alpha, 2\alpha}$ so that $\varphi_\alpha \equiv 1$ in Δ_α and $\varphi_\alpha \equiv 0$ in $X \setminus \Delta_{2\alpha}$ with the property that

$$|\widehat{\Lambda^4 \varphi_\alpha}(\xi)| \leq c,$$

with c a constant independent of α .

From above we have

$$(5.5) \quad \|f\|_{W^\epsilon(X)} \leq \|f\|_{W^\epsilon(X \setminus \Delta_\alpha)} + \|\varphi_\alpha f\|_{W^\epsilon(X)}.$$

For the second term on the right hand side we can consider f and φ_α to be defined on all of \mathbb{C}^2 by extension by 0 outside of B . We then use

$$(5.6) \quad \begin{aligned} \|\varphi_\alpha f\|_{W^\epsilon(X)} &\leq \|f \Lambda^\epsilon \varphi_\alpha\|_{L^2(X)} + \|\varphi_\alpha \Lambda^\epsilon f\|_{L^2(X)} \\ &\leq \|f \Lambda^\epsilon \varphi_\alpha\|_{L^2(X)} + c(\alpha) \|f\|_{W^\epsilon(X)}, \end{aligned}$$

where $c(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0$.

We concentrate on the term $\Lambda^\epsilon \varphi_\alpha = \Lambda^{\epsilon-4} \circ \Lambda^4 \varphi_\alpha$.

$$\begin{aligned} |\widehat{\Lambda^\epsilon \varphi_\alpha}| &= (1 + |\xi|^2)^{(\epsilon-4)/2} |\widehat{\Lambda^4 \varphi_\alpha}| \\ &\leq c(1 + |\xi|^2)^{(\epsilon-4)/2}, \end{aligned}$$

from which we have $\widehat{\Lambda^\epsilon \varphi_\alpha} \in L^p(\mathbb{R}^4)$ for $p > 4/(4 - \epsilon)$ and $\Lambda^\epsilon \varphi_\alpha \in L^q(\mathbb{R}^4)$ for $q < 4/\epsilon$. Hence we have

$$\|f \Lambda^\epsilon \varphi_\alpha\|_{L^2(X)} \lesssim \left(\int_X |f|^{2s'} dV_X \right)^{\frac{1}{2s'}} \left(\int_{\mathbb{R}^4} |\Lambda^\epsilon \varphi_\alpha|^{2s} \gamma^4 d\mathbf{x}^4 \right)^{\frac{1}{2s}},$$

where $s = q/2$ and $s' = q/(q - 2)$, i.e.

$$\|f \Lambda^\epsilon \varphi_\alpha\|_{L^2(X)} \lesssim \|f\|_{L^p(X)}, \quad p > \frac{4}{2 - \epsilon}.$$

Returning to (5.6), we have for some α small enough (which we fix from now on):

$$\begin{aligned} \|\varphi_\alpha f\|_{W^\epsilon(X)} &\lesssim \|f \Lambda^\epsilon \varphi_\alpha\|_{L^2(X)} + (s.c.) \|f\|_{W^\epsilon(X)} \\ &\lesssim \|f\|_{L^p(X)} + (s.c.) \|f\|_{W^\epsilon(X)}. \end{aligned}$$

If we assume $f \in L^p_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$, then $\|\xi_{-\epsilon} f\|_{L^2(X)} \lesssim \|f\|_{L^p(X)}$ for $p > 4/(2 - \epsilon)$ and $f \in L^{2,-\epsilon}_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. Furthermore, $\gamma^{1-\epsilon} f \in L^{2,-1}_{(0,1)}(X)$, and we can apply Theorem 5.1 to show $\gamma^{1-\epsilon} f \in W^1(X)$. Since $f \in L^{2,-\epsilon}(X)$, we also have $f \in W^{1,1-\epsilon}(X)$ and we can apply (5.4) to estimate the term $\|f\|_{W^\epsilon(X \setminus \Delta_\alpha)}$ in (5.5). Finally, we can write

$$\begin{aligned} \|f\|_{W^\epsilon(X)} &\lesssim \|f\|_{W^\epsilon(X \setminus \Delta_\alpha)} + \|f\|_{L^p(X)} + (s.c.) \|f\|_{W^\epsilon(X)} \\ &\lesssim \|f\|_{W^{1,1-\epsilon}(X)} + \|f\|_{L^p(X)} + (s.c.) \|f\|_{W^\epsilon(X)}. \end{aligned}$$

We conclude the following

Proposition 5.2. *Let $0 \leq \epsilon \leq 1$ and $p > 4/(2 - \epsilon)$. For $f \in L^p_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with support in a neighborhood of the singularity then $f \in W^\epsilon(X)$. The norm $\|f\|_{W^\epsilon(X)}$ is bounded by*

$$\|f\|_{W^\epsilon(X)} \lesssim \|f\|_{W^{1,1-\epsilon}(X)} + \|f\|_{L^p(X)}.$$

6. SUBELLIPTIC ESTIMATES

We apply Proposition 5.2 to each component, f_j , $j = 1, 2$, of a given form $f \in L^p_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with support in a neighborhood of the singularity. To bound the terms

$$\int_X \gamma^{2-2\epsilon} |\nabla f_j|^2 dV$$

we use

$$\begin{aligned} \int_X \gamma^{2-2\epsilon} |\nabla f|^2 dV &\lesssim \|\gamma^{2-\epsilon} \bar{L} f\|^2 + \|\gamma^{2-\epsilon} L f\|^2 \\ &\lesssim \|\gamma \bar{L} (\gamma^{1-\epsilon} f)\|^2 + \|\gamma L (\gamma^{1-\epsilon} f)\|^2 + \|\xi_{-\epsilon} f\|^2. \end{aligned}$$

Using the estimate $\|\xi_{-\epsilon} f\|_{L^2(X)} \lesssim \|f\|_{L^p(X)}$ for $p > 4/(2 - \epsilon)$ and Proposition 5.2 we have

$$\|f\|_{W^\epsilon(X)}^2 \lesssim \|\gamma \bar{L} (\gamma^{1-\epsilon} f)\|^2 + \|\gamma L (\gamma^{1-\epsilon} f)\|^2 + \|f\|_{L^p(X)}^2$$

As in Section 5 above, given $0 \leq \epsilon \leq 1$ and a form $f \in L^p_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ for $p > 4/(2 - \epsilon)$, we have $\gamma^{1-\epsilon}f \in L^{2,-1}_{(0,1)}(X)$, and we can apply Theorem 5.1 to the form $\gamma^{1-\epsilon}f$ in place of f to conclude

$$(6.1) \quad \begin{aligned} \|f\|_{W^\epsilon(X)}^2 &\lesssim \|\gamma \bar{\partial}(\gamma^{1-\epsilon}f)\|^2 + \|\gamma \bar{\partial}^*(\gamma^{1-\epsilon}f)\|^2 + \|f\|_{L^p(X)}^2 \\ &\lesssim \|\gamma^{2-\epsilon} \bar{\partial}f\|^2 + \|\gamma^{2-\epsilon} \bar{\partial}^*f\|^2 + \|f\|_{L^p(X)}^2. \end{aligned}$$

We can now use a partition of unity to remove the assumption of support near the singularity. We use estimates given by (6.1) for forms with support in a neighborhood of the singularity, and subelliptic $1/2$ -estimates for forms with support near the boundary ∂X (Theorem 5.1.2 [CS]):

$$(6.2) \quad \|f\|_{W^\epsilon(X)} \lesssim \|f\|_{W^{1/2}(X)} \lesssim \|\bar{\partial}f\| + \|\bar{\partial}^*f\|, \quad 0 \leq \epsilon \leq 1/2.$$

Combining (6.1) and (6.2), we arrive at our subelliptic estimate which we write in the form of

Theorem 6.1. *Let $0 \leq \epsilon \leq 1/2$, $p > 4/(2 - \epsilon)$, and $f \in L^p_{(0,1)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$. Then the following subelliptic estimate holds*

$$\|f\|_{W^\epsilon(X)} \lesssim \|\bar{\partial}f\| + \|\bar{\partial}^*f\| + \|f\|_{L^p(X)}.$$

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